Eliciting Beliefs from Risk Averse Forecasters Using a Log Scoring Rule

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"*A probability is a price in a manner of speaking*" L J Savage (JASA,1971)

1 Introduction

Coherent probabilities are prices. De Finetti said it. Savage said it. And A.P.Dawid said it again at the Valencia 7 workshop an the foundations of Bayesian statistics, using the De Finetti elicitation game to explain the logic of measuring uncertain personal knowledge. Yet how many of the sophisticated inference models articulated during the scientific programme of the conference were put to an elicitation test, in whole or in part? None that I know of. Probabilities as personal prices may be foundational, but practically speaking they remained there, in the foundations, during Valencia 7.

One supply side explanation of this situation is that credible, effective elicitation methods are hard to devise and costly to implement. For example, as Ed Leamer explained in Valencia 3, the De Finetti Dutch book argument for finitely additive probability doesn't work as an operational elicitation mechanism. "The two defects of de Finetti's elicitation game are that you may refuse to play, and when you do play I cannot expect to elicit your true probability" (Leamer (1986,223). Strategic considerations in the bargaining process tend to make *observed* transaction prices different from the elicitee's *private* marginal valuations that define her personal, coherent probabilities. Leamer went on to explain several other practical methods for eliciting beliefs including: (1) a Marschak-Vickery second-price sealed-bid auction for contingent commodities where the seller's prices are unknown to and non-informative for the elicitee, and (2) Savage's development of proper scoring rule games where an elicitee effectively acts as a profit maximising, perfectly discriminating monopolist/monopsonist in markets whose net demand schedules are implicitly defined by a scoring rule. But all of these elicitation methods presume risk neutrality or linear utility in the payoff medium. As Savage argued, "the need to avoid the distortion of response induced by the non-linearity of the respondent's utility" (Savage (1971,p798)) is yet another important, practical limitation on using scoring rule type elicitation games. Risk averse agents maximize *expected utility* not *expected wealth*.

Our paper shows that one particular proper scoring rule, the Log scoring rule, can be augmented to avoid the distortions introduce by non-linear utility (risk neutrality) for a limited class of agents, constant absolute risk averse (CARA) agents with negative exponential utility of wealth function $u(z)=-e^{-\frac{1}{\tau} z}$. The general form of the conventional Log scoring rule is A•Ln($r(s)$)+$b(s)$ for a reported pmf ***r(s)***={p(1)...p(n)} on n discrete states S from the set {1,...n} with A>0 and $b(s)$ some arbitrary function on states (Proposition 2.29, Bernardo and Smith (1994,p. 73)). We call Ln(*r(s))* the *simple* Log score rule for a reported pmf ***r(s)***. While the Log scoring rule is "proper" (honest reporting is expected wealth maximizing), it is not expected utility maximizing for a risk averse elicitee.The augmentation we propose to correct this problem is based on the following reward function: m+ $\tau\{D(\boldsymbol{\pi(s)}\|\boldsymbol{r(s)}) - \text{Ln}\ \frac{\pi(s)}{r(s)}\}$ where $D(\boldsymbol{\pi(s)}\|\boldsymbol{r(s)})=\Sigma_S\ \pi(s)\,\text{Ln}\left(\frac{\pi(s)}{r(s)}\right)$ is Kullback Leibler (KL) discrepancy (note that our symbol for KL discrepancy $D(\boldsymbol{\pi}\|\boldsymbol{r})$ follows Cover and Thomas (1991,p.18) which is just a permutation of the variable symbols $\delta(\boldsymbol{r}|\boldsymbol{\pi})$ in Bernardo and Smith (1994,p. 78)). The elicitor specifies m a positive constant and ***π(s)*** a pmf on S. The elicitee responds with a report τ for her risk tolerance and a report ***r(s)*** for her beliefs. Hence the augmented Log scoring rule elicits information on both beliefs and risk attitude *simultaneously*. We show in section 3 that any CARA elicitee maximizes her expected utility reporting her true beliefs and her true risk tolerance.

The augmented Log scoring rule has both a "statistical" interpretation and an "economic" interpretation. The KL discrepancy term in the payoff function, $D(\boldsymbol{\pi(s)}\|\boldsymbol{r(s)})=\Sigma_S\ \pi(s)\,\text{Ln}\left(\frac{\pi(s)}{r(s)}\right)$, is an average at the elicitor's pmf ***π(s)*** of the simple Log score differences $\text{Ln}\left(\frac{\pi(s)}{r(s)}\right)$, also know as "weights of evidence" and Log Bayes factors (Kass and Raftery(1995)), in the elicitors favour. As long as $\boldsymbol{\pi(s)}\neq\boldsymbol{r(s)}$, $D(\boldsymbol{\pi(s)}\|\boldsymbol{r(s)})>0$, the elicitor expects to out score the elicitee using the simple Log scoring rule by the positive amount $D(\boldsymbol{\pi(s)}\|\boldsymbol{r(s)})$ on average. The augmented Log scoring rule "credits" this average (after scaling by τ) to the elicitee and then rewards (penalizes) reports in state S=*s* that make the simple Log score difference $\text{Ln}\ \frac{\pi(s)}{r(s)}$ below (above) the elicitor's average $D(\boldsymbol{\pi(s)}\|\boldsymbol{r(s)})$. In effect the elicitee gets rewarded or penalized in terms of her ability to predict *relatively* better or worse than the elicitors average Log score difference indicated by a benchmark pmf ***π(s)*** specified publicly in advance by the elicitor. As explained in section 2, $\tau D(\boldsymbol{\pi(s)}\|\boldsymbol{r(s)})$ is also the economic surplus (gains from trade) a CARA agent with beliefs ***r(s)*** and risk tolerance τ can expect trading in competitive contingent claims markets with prices ***π(s)***. The elicitee's self reports τ and ***r(s)*** determine the magnitude of this surplus but at the same time put it, and her endowment of wealth m, at risk by committing her to risky transactions $\tau\text{Ln}\ \frac{r(s)}{\pi(s)}$ in each state *s.*

The outline of the remainder of the paper is as follows. Section 2 of the paper develops an explanation of the conventional Log scoring rule as a 2 person general equilibrium game of exchange between an elicitor and an elicitee. The elicitor has an incompletely specified mechanism design problem. He needs to provide an elicitee with incentives to participate, rewards for behaving honestly, and credibility about his ability to make contracted payments. A credible Log scoring rule must address these questions. Section 3 of the paper uses the general equilibrium pricing formula in the conventional Log scoring rule elicitation game when the elicitee is also a CARA agent to show how to "correct" the reporting biases and induce honest revelation of beliefs using the augmented Log scoring rule. Section 4 examines a potentially useful application : What can the Log scoring rule, augmented or conventional, expect to elicit from a group of risk averse forecasters? Section 5 has a discussion of some limitations (What happens if the elicitee isn't CARA? What range of beliefs can credibly be elicited?) and a brief conclusion.

■ 2 The Log scoring rule as a 2 person general equilibrium game of exchange

Consider a CARA agent with negative exponential utility of wealth function u(z)=$-e^{-\frac{1}{\tau}z}$ and beliefs described by a coherent pmf (probability mass function) $\boldsymbol{q(s)}$={q(1)...q(n)} for a discrete uncertain state variable S with values $s\epsilon${1,...n}. Following Lad(1996,42-45) we define the *constituent events* for S as the logical truth value of propositions of the form "S= s", denoted by the assertion in brackets (S= s). Each (S= s), is an uncertain 0/1 valued quantity used to define tradeable units of contingent wealth (or some other desireable, transferable payoff medium) paying \$1 if the proposition "S= s" is true and \$0 otherwise. Assume for the moment that the CARA agent can buy and sell unit contingent wealth claims (S= s), $s\epsilon${1,...n}, in markets as a price taker subject to a budget constraint $\boldsymbol{p \bullet \omega} = \boldsymbol{p \bullet z}$, where $\boldsymbol{z(s)}$={z(1)...z(n)} is a non-negative vector of her contingent wealth $z(s)$ in each state s, $\boldsymbol{\omega(s)}$={ω(1)....ω(n)} is the agent's non-negative endowment $\omega(s)$ in each state s, and $\boldsymbol{p(s)}$={p(1)...p(n)} is a set of non-negative prices $p(s)$ for unit contingent wealth claims (S= s). Prices $\boldsymbol{p}$ must satisfy the constraint $\sum_s p(s)=1$, otherwise arbitrage can create Dutch books against the market. In most cases in this paper the endowment vector $\boldsymbol{\omega(s)}$ is an n-tuple of constants {m,m,....m} denoted by m , so that the budget constraint becomes $\boldsymbol{p \bullet z}=m$. In our notation, n-tuples like $\boldsymbol{q(s)}$, $\boldsymbol{\omega(s)}$ etc are indicated by bold italics, we typically suppress arguments when they can be readily inferred from the context, eg writing $\boldsymbol{q}$ for $\boldsymbol{q(s)}$, and $\boldsymbol{x \bullet y}$ indicates the conventional inner product $\sum_s$ x(s)y(s).

Instead of maximizing expected utility, EU(τ,$\boldsymbol{q}$,$\boldsymbol{z}$)≡Σ_s $q(s)\left(-e^{-\frac{1}{\tau}z(s)}\right)$, subject to a budget constraint, consider the dual problem 1.1, choosing non-negative contingent wealth levels $\boldsymbol{z(s)}$ to minimize expenditure $\boldsymbol{p \bullet z}$ at market prices subject to an expected utility or certainty equivalent constraint, where certainty equivalents are CE(τ,$\boldsymbol{q}$,$\boldsymbol{z^o}$)≡$-\tau$Ln(–EU(τ,$\boldsymbol{q}$,$\boldsymbol{z^o}$)):

1.1• $\underset{\boldsymbol{z}}{\text{Min}}\ \boldsymbol{p \bullet z}$ subject to $\Sigma_s\ q(s)\left(-e^{-\frac{1}{\tau}z(s)}\right)=eu$ or to $ce=-\tau\text{Ln}(eu)$

The first order necessary (and in this case sufficient) conditions for 1.1 are, for each s:

1.2• $p(s) = \lambda\, q(s)\, \frac{1}{\tau}\, e^{-\frac{1}{\tau}z(s)}$,where $\lambda=\frac{\tau \sum p(t)}{(-eu)}=\frac{\tau}{-eu}$ when $\sum_t p(t)$=1.

Solving for the n expenditure minimizing (compensated) demands $\boldsymbol{z}^c$ for wealth in state s, we find:

1.3 $z^c(s,\boldsymbol{p},\boldsymbol{q},\tau,ce)=ce+\tau\text{Ln}(\frac{q(s)}{p(s)})$ valid for $\boldsymbol{z}^c\geq 0$

The compensated demand function 1.3, $ce+\tau\text{Ln}(\frac{q(s)}{p(s)})$, has the form of a general Log scoring rule for the agent's pmf $\boldsymbol{q(s)}$, ALog($q(s)$)+$b(s)$ (Proposition 2.29, Bernardo and Smith (1994,p. 73)) and any general Log scoring rule can be expressed in the form 1.3. Optimal wealth levels in state s are directly proportional to the differences in simple Log scores Ln($\frac{q(s)}{p(s)}$). When the agent anticipates a higher (lower) simple Log score than the market, Ln($\frac{q(s)}{p(s)}$)>(<)0, then her optimal decision net of her certainty equivalent, $z^c(s)-ce$, is to purchase (sell) contingent wealth in state s in amounts proportional to the Log score difference Ln($\frac{q(s)}{p(s)}$)in that state, with her risk tolerance τ the factor of proportionality.

While *actual* Log score differences determine optimal choices, *expected* Log score differences, Kullback Leibler discrepancies, determine the valuations placed on these optimal choices. First, taking market expectations of optimal wealth, $\boldsymbol{p}\bullet\boldsymbol{z^c} = \Sigma_S\, p(s) z^c(s,\boldsymbol{p},\boldsymbol{q},\tau,ce)$, we obtain the expenditure function, the minimum value of 1.1 :

1.4• $m(\boldsymbol{p},\boldsymbol{q},\tau,ce) = ce - \tau\ \mathrm{D}(\boldsymbol{p}\|\boldsymbol{q})$

where $\mathrm{D}(\boldsymbol{p}\|\boldsymbol{q}) = \Sigma_S\, p_S\ \mathrm{Ln}\ \frac{p(s)}{q(s)}$ is KL discrepancy, between $\boldsymbol{p}$ and $\boldsymbol{q}$ (in that order). Next, taking subjective expectations of optimal wealth, $\boldsymbol{q}\bullet\boldsymbol{z^c} = \Sigma_S\, q(s) z^c(s,\boldsymbol{p},\boldsymbol{q},\tau,ce)$ we have :

1.5• $\boldsymbol{q}\bullet\boldsymbol{z^c}(\boldsymbol{p},\boldsymbol{q},\tau,ce) = ce + \tau\ \mathrm{D}(\boldsymbol{q}\|\boldsymbol{p})$

where $\mathrm{D}(\boldsymbol{q}\|\boldsymbol{p}) = \Sigma_S\, q_S\ \mathrm{Ln}\left(\frac{q(s)}{p(s)}\right)$ is KL discrepancy, between $\boldsymbol{q}$ and $\boldsymbol{p}$ (in the reverse order to 1.4).

The functions $z^c(s,\boldsymbol{p},\boldsymbol{q},\tau,ce)$ in 1.3 are called *compensated* demand functions to differentiate them from *ordinary* market demand functions $z^o(s,\boldsymbol{p},\boldsymbol{q},\tau,m)$ that maximize expected utility $\Sigma_S\, q(s)\left(-e^{-\frac{1}{\tau} z(s)}\right)$ subject to a budget constraint $\boldsymbol{p}\bullet\boldsymbol{z}=m$, where m is income in the same non contingent units of wealth as prices. Standard duality arguments (Cornes(1992)) ensure that for all s, $z^c(s,\boldsymbol{p},\boldsymbol{q},\tau,ce) = z^o(s,\boldsymbol{p},\boldsymbol{q},\tau,m(s,\boldsymbol{p},\boldsymbol{q},\tau,ce))$ and $z^o(s,\boldsymbol{p},\boldsymbol{q},\tau,m) = z^c(s,\boldsymbol{p},\boldsymbol{q},\tau,\mathrm{CE}(\tau,\boldsymbol{q},\boldsymbol{z^o}(\boldsymbol{p},\boldsymbol{q},\tau,m)))$. That is, for any price $\boldsymbol{p}$, compensated demand functions at ce are ordinary demand functions evaluated at the expenditure minimizing income for ce and ordinary demand functions at income m are compensated demand functions evaluated at certainty equivalent levels for the ordinary demand at income m. These ordinary demand functions, which will serve as the basis for an augmented Log scoring rule (explained below), can be obtained from 1.3 by using 1.4 to substitute out $ce = m + \tau\mathrm{D}(\boldsymbol{p}\|\boldsymbol{q})$:

1.6 $\boldsymbol{z^o}(s,\boldsymbol{p},m,\boldsymbol{q},\tau) = m + \tau\{\mathrm{D}(\boldsymbol{p}\|\boldsymbol{q}) + \mathrm{Ln}\ \frac{q(s)}{p(s)}\}$

For any two pmfs $\boldsymbol{a}$, $\boldsymbol{b}$, KL discrepancy $\mathrm{D}(\boldsymbol{a}\|\boldsymbol{b}) \geq 0$ with equality iff $a(s)=b(s)$ for all states s. Hence the scaled KL discrepancy $\tau\ \mathrm{D}(\boldsymbol{p}\|\boldsymbol{q})$ in 1.4 can be interpreted as the *cost savings* available to the agent from trade at prices $\boldsymbol{p}$. The greater the KL discrepancy $\mathrm{D}(\boldsymbol{p}\|\boldsymbol{q})$ between prices and beliefs the cheaper it is for an agent to obtain any target level of certainty equivalent/expected utility, the more so the more risk tolerant they are. When prices $\boldsymbol{p}$ are equal to beliefs $\boldsymbol{q}$, *fair odds* prices, no costs savings are available since $\mathrm{D}(\boldsymbol{q}\|\boldsymbol{q})=0$ and no trade takes place (1.3). Cost savings $\tau\mathrm{D}(\boldsymbol{p}\|\boldsymbol{q})$ are bounded above by $\mathrm{Ln}(q(s_*))$, achieved in principle when the market believes that one of the constituent events ranked least likely by the agent, $S=s_*$ where $s_* \epsilon\ \mathrm{Argmin}_s((q(s))$, is a certainty and prices $(S=s_*)$ and other contingencies accordingly.

The cost savings available from trading opportunities at prices $\boldsymbol{p}$ have a natural (to an economist) interpretation as economic surplus, gains from trade measured by Hicksian compensating and equivalent variations in income (see Fountain (2002) for further details). In particular scaled KL discrepancy $\tau\ \mathrm{D}(\boldsymbol{p}\|\boldsymbol{q})$ is the maximum amount the CARA agent Q is willing to pay to face market prices $\boldsymbol{p}$ rather than face prices equal to her beliefs $\boldsymbol{q}$. Q's positive willingness to pay to accept wealth risks at market prices $\boldsymbol{p}$ different from beliefs $\boldsymbol{q}$ is an incentive for someone else with power to determine prices $\boldsymbol{p(s)}$ to willingly engage in trade with Q, helping to solve the participation problem identified by Leamer, at least on the demand side, ie what can the elicitee expect to get out of this game. To see this note that 1.4, 1.5, and 1.6 combine to create particularly simple certainty equivalent valuation formulae for CARA agents. The certainty equivalent for a CARA agent with beliefs ***q*** and risk tolerance τ_Q trading at prices ***p*** with income *m* can be expressed in two equivalent forms as a function of budget set parameters:

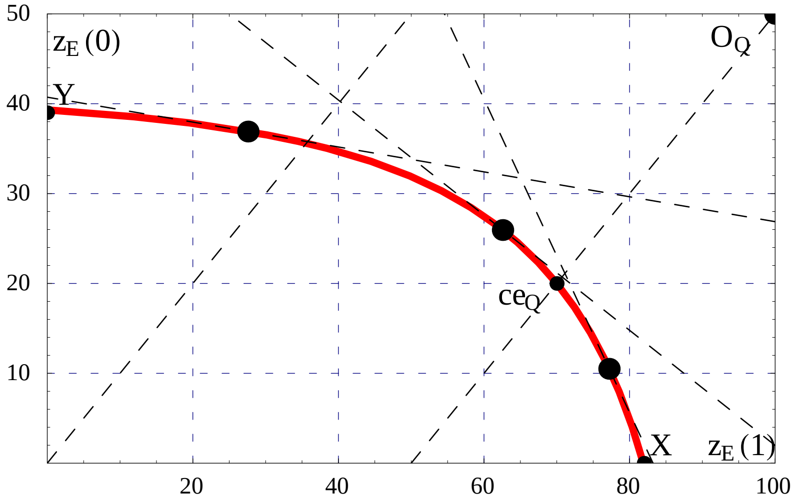
1.7 $$\begin{aligned} \mathrm{CE}(\boldsymbol{p}, m, \boldsymbol{q}, \tau) &= m + \tau\{\mathrm{D}(\boldsymbol{p}||\boldsymbol{q}) \\ &= \boldsymbol{q}\bullet\boldsymbol{z}^o(\boldsymbol{p}, m, \boldsymbol{q}, \tau) - \tau\, \mathrm{D}(\boldsymbol{q}||\boldsymbol{p}). \end{aligned}$$

The top equation in 1.7 says trading opportunities are valued in terms of market wealth *m plus* economic surplus $\tau\mathrm{D}(\boldsymbol{p}||\boldsymbol{q})$ while the bottom equation says that trading opportunities are valued in terms of subjectively expected wealth $\boldsymbol{q}\bullet\boldsymbol{z}^o(\boldsymbol{p}, m, \boldsymbol{q}, \tau)$ *minus* an adjustment for risk, the indirect Pratt Arrow risk premium $\tau\ \mathrm{D}(\boldsymbol{q}||\boldsymbol{p})$, (see Fountain (2002) for further details). Knowing these facts about the implicit agent they are dealing with may encourage elicitees to participate in a Log scoring rule game.

Suppose an elicitor presents an elicitee with an oppportunity to trade against a CARA agent Q with beliefs ***q(s)*** and risk tolerance τ_Q. Q trades at prices ***p(s)*** announced by the elicitee using the compensated demand functions 1.3 for a given certainty equivalent ce_Q. This trading arrangement is equivalent to a two part tariff where the elicitee announces prices ***p(s)*** to an agent Q with an income ce_Q intially facing prices equal to her beliefs, meets Q's *ordinary* market demands as per 1.6, and extracts the surplus $\tau\mathrm{D}(\boldsymbol{p}||\boldsymbol{q})$ that Q gains from trade as a "fee" in the form of a change in Q's income. But outcomes are uncertain and naive application of 1.3 may expose the elicitor or the elicitee to large, and frankly unaffordable, negative wealth levels. To avoid these problems, the elicitor can ensure that trades only occur at non-negative wealth levels for both the elicitee and Q up to a pre-defined limit of aggregate wealth resources ϖ(s) in each state S=*s*. The n-tuple ***ϖ(s)*** is "money on the table" credibly specified by the elicitor. The elicitee acts as a *residual claimant* receiving $\varpi(s)-ce+\tau_Q\text{-}\mathrm{Ln}(\frac{p(s)}{q(s)})\geq 0$ up to the feasible limit ϖ(s). Notice that the elicitee's contingent wealth payoff function $\varpi(s)-ce_Q+\tau_Q\,\mathrm{Ln}(\frac{p(s)}{q(s)})$ has the general form of the Log scoring rule *A*Log(*p(s))*+*b(s)* (Proposition 2.29, Bernardo and Smith (1994,p. 73)) in her reported prices ***p(s)***.The elicitor controls all other aspects of the payoff function.

The Edgeworth box in Figure 1 illustrates the situation for a two state $s\epsilon\{0,1\}$ example. The dimensions of the box $(0,\varpi(s))$ in each state specify the aggregate wealth available for the elicitee and elicitor to exchange in each contingency, ***ϖ(s)***=($100,50) in the figure. Measuring Q's wealth from the upper right hand corner, O_Q the curve $\mathrm{Y}ce_Q\mathrm{X}$, presents a typical constraining indifference curve ($ce\text{-}\tau_Q\,\mathrm{Ln}(\frac{p(s)}{q(s)})$ in each state *s* from 1.3) chosen by the elicitor , where ce_Q=\$30 in in the figure. The elicitee's residual wealth in each state $z_E(s)=\varpi(s)-ce_Q+\tau_Q\,\mathrm{Ln}(\frac{p(s)}{q(s)})$ is measured from the lower left hand corner O_E. Q's indifference curve $\mathrm{Y}ce_Q\mathrm{X}$ is the constraint faced by the elicitee. The dashed tangent lines in Fig 1 represent alternative possible prices ***p(s)*** in the form of budget lines the elicitee might offer Q, with the corresponding tangent points on Q's indifference curve indicating Q's optimal response and hence the elicitee's residual wealth.

Fig 1



Let the elicitee's beliefs about states be represented by the pmf ***b(s)***. Her expected wealth when specifying prices ***p(s)*** prices is $\Sigma_s\, b(s)\{\varpi(s) – ce_Q + \tau_Q \,\mathrm{Ln}(\frac{p(s)}{q(s)})\ \}$=$\boldsymbol{b} \bullet \boldsymbol{\varpi} – ce_Q + \tau_Q\, \Sigma_s\, b(s) \mathrm{Ln}(\frac{b(s)\, p(s)}{q(s)\, b(s)})$= $\boldsymbol{b} \bullet \boldsymbol{\varpi} – ce_Q + \tau_Q \{\mathrm{D}(\boldsymbol{b}||\boldsymbol{q}) – \mathrm{D}(\boldsymbol{b}||\boldsymbol{p})\}$. Since $\mathrm{D}(\boldsymbol{b}||\boldsymbol{p}) \geq 0$, any difference between the prices she sets and her beliefs can only reduce her expected wealth in proportion to the KL discrepancy between her announced prices and her beliefs. Hence honest reporting ***p***=***b*** is expected wealth maximizing, at least whenever the non negativity and feasibility constraints are met strictly. (In Figure 1 the limits to honest reporting for expected wealth maximizing elicitee's involve tangent lines to Q's indifference curve Yce_QX at "y" and at "x" on the y-axis and x-axis respectively, a point we take up in the final section.)

But risk averse agents maximize expected *utility*, not expected *wealth*. In general one would expect an elicitee fully informed about this elicitation game to set prices so as to extract maximum benefits for herself given the constraining indifference curve she faces. Standard Pareto efficiency arguments in economics imply that when she does so the two agents' marginal rates of substitution (MRS) for contingent wealth must be equal for wealth in all states, at least at an interior solution given the feasibility constraints.Taking state 1 as a numeraire and letting ρ(s) be the absolute value of that common MRS between wealth in state *s* and wealth in state 1, the pmf $\pi(\mathrm{s}) = \frac{\rho(s)}{\Sigma_t\, \rho(t)}$ defines a market pmf that when reported by the elicitee will lead the elicitor responding according to 1.3 an expected utility maximizing choice for the elicitee.

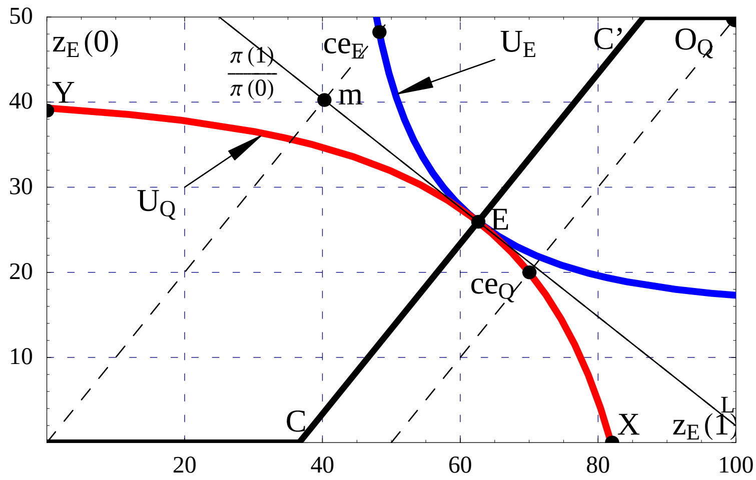
Suppose the elicitee is also a CARA agent, with beliefs ***b(s)*** and risk tolerance τ_b. The common marginal rate of substitution or *equilibrium odds* $\frac{\pi(s)}{\pi(t)}$, $\Sigma_s\, \pi(s)=1$, between pairs of states *s,t* takes a particularly simple Log-odds form:

1.8 $$\mathrm{Ln}\frac{\pi(s)}{\pi(t)} = \frac{\tau_b}{\tau_b+\tau_Q}\,\mathrm{Ln}\frac{b(s)}{b(t)} + \frac{\tau_Q}{\tau_b+\tau_Q}\,\mathrm{Ln}\frac{q(s)}{q(t)} - \frac{1}{\tau_b+\tau_Q}\,[\varpi(s)\text{-}\varpi(t)]$$

The expected utility maximizing prices ***π(s)*** the elicitee announces as her beliefs are, in Log-odds form, a weighted average of the Log-odds beliefs of the two trading agents, adjusted for any aggregate wealth risk in the two states [$\varpi(s)$-$\varpi(t)$]. The weights on each agent's beliefs are their relative risk tolerances, $\frac{\tau_b}{\tau_b+\tau_Q}$ and $\frac{\tau_Q}{\tau_b+\tau_Q}$. An expected utility maximizing CARA elicitee playing a Log scoring rule game strategically reports *competitive equilibrium, Pareto efficient prices* in the elicitation game 1.8, not her honest beliefs.

Figure 2 illustrates the argument in the 2 state model used for Figure 1. The indifference curve U_E is tangent to Q's constraining indifference curve Yce_QX at the elicitee's optimal choice E. The absolute value of the slope, $\frac{\pi(1)}{\pi(0)}$, of the straight line mEL tangent at E defines the general competitive equilibrium relative prices or equilibrium odds for contingent claims that the elicitee should report strategically as her beliefs. In this case equlbrium odds are neither the odds of the elicitor's agent Q, $\frac{q(1)}{q(0)}$ (slope of Q's indifference curve U_Q at CE$_Q$) nor the odds of the elicitee , $\frac{b(1)}{b(0)}$ (slope of the elicitee's indifference curve U_E at CE$_E$). Honest reporting of beliefs is not expected utility maximizing. Pareto efficient pricing is.

Figure 2



The basic intuition here is straightforward. A risk averse elicitee responding to a request to report her beliefs will anticipate what her reported beliefs will be used for and what the wealth consequences will be for herself. The elicitation game in Figure 2 appears attractive overall if the elicitee's outside options (ie not participating) are worth less than CE$_Q$. The game basically involves extracting surplus from the elicitor while receiving protection against negative wealth. But participating exposes her to wealth risks. Reporting beliefs honestly will maximize the elicitee's expected wealth at too high an exposure to wealth risks for a risk averse elicitee. Therefore she reports strategically, not honestly. Reporting beliefs that in odds form are general competitive equilibrium relative prices supporting her expected utility maximizing wealth allocation will do the job.

■ 3 Augmenting the Log scoring rule to avoid strategic reporting biases

This behavioural model of the Log scoring rule as a 2-person equilibrium game of exchange provides insights into how the elicitor can "correct" strategic reporting biases. In this case, practically speaking, it means taking away the elicitee's market power. Suppose the Log scoring rule $\varpi(s) - ce_Q + \tau_Q \,\mathrm{Ln}(\frac{p(s)}{q(s)})$ facing the elicitee is replaced by an ordinary demand function, 1.6, for the elicitee as a CARA agent. The elicitor specifies prices $\pi(s)$ for contingent claims and asks the elicitee to respond with *both* a risk tolerance τ and a coherent probability report $\boldsymbol{p(s)}$, paying her in return:

1.9 $\quad z_E^O(s,\boldsymbol{\pi},m,\boldsymbol{p},\tau) = m + \tau\{\mathrm{D}(\boldsymbol{\pi}||\boldsymbol{p}) + \mathrm{Ln}\,\frac{p(s)}{\pi(s)}\}$

The payoff function 1.9 , with one term linear in the Log of the elicitee's probability report $p(s)$, has the form of an *augmented* Log scoring rule. One part, $m + \tau\mathrm{D}(\boldsymbol{\pi}||\boldsymbol{p})$, is independent of the state S=*s*, but depends on the elicitee's probability report $\boldsymbol{p}$ through KL discrepancy. Reports $\boldsymbol{p}$ that are farther in KL discrepancy $\mathrm{D}(\boldsymbol{\pi}||\boldsymbol{p})$ from the elicitor's prices $\boldsymbol{\pi}$ earn the elicitee more sure money generally (across all states) in the augmented elicitation game, the more so the reported risk tolerance τ. As we noted above, $\tau\mathrm{D}(\boldsymbol{\pi}||\boldsymbol{p})$, is a measure of economic surplus, in this case the surplus of CARA agent with beliefs $\boldsymbol{p}$ and risk tolerance τ trading in markets with prices $\boldsymbol{\pi}$. Hence the first part of the augmented scoring rule, $m + \tau\mathrm{D}(\boldsymbol{\pi}||\boldsymbol{p})$, can be regarded as making a payment to the elicitee of a certainty equivalent, income plus self reported surplus (1.7), for the trading opportunity $(\boldsymbol{\pi},m)$ to a CARA agent with beliefs $\boldsymbol{p}$ and risk tolerance τ. The second part of the payoff function is simply the scaled difference in simple Log scores: $\tau\mathrm{Ln}(\frac{p(s)}{\pi(s)})$. This term is positive (negative) when the elicitee predicts S=*s* *relatively* better (worse) using *p(s)* than the elicitor can using π(s), where better or worse prediction is assessed by the simple Log scoring rule. The elicitee effectively chooses through her report $\boldsymbol{p}$ the surplus $\tau\mathrm{D}(\boldsymbol{\pi}||\boldsymbol{p})$ she would earn as a CARA agent in a competitive market at prices $\boldsymbol{\pi}$ but simultaneously puts it and her endowment of income *m* at risk by undertaking to accept risky transactions $\tau\ \mathrm{Ln}\,\frac{p(s)}{\pi(s)}$ in each state *s*.

An alternative, complementary "statistical" interpretation of 1.9 focusses on the predictive aspects of the problem. Rewrite $\mathrm{D}(\boldsymbol{\pi}||\boldsymbol{p}) + \mathrm{Ln}\,\frac{p(s)}{\pi(s)}$ as $\mathrm{D}(\boldsymbol{\pi}||\boldsymbol{p}) - \mathrm{Ln}\,\frac{\pi(s)}{p(s)}$ and note that $\mathrm{D}(\boldsymbol{\pi}||\boldsymbol{p}) = \Sigma\pi(s)\mathrm{Ln}\,\frac{\pi(s)}{p(s)} \geq 0$ is an *average* amount by which an elicitor announcing prices $\boldsymbol{\pi}$ expects to predict better than an elicitee reporting $\boldsymbol{p}$ using the simple Log scoring rule. The payoff structure 1.9 effectively provides the elicitee with a fixed income *m* augmented by a positive incremental reward in state S=s if, through her report *p(s)*, she can keep the elicitor below his average $\mathrm{D}(\boldsymbol{\pi}||\boldsymbol{p})$ predictive advantage, but a negative incremental reward, a penalty, if the elicitor does better than his average predictive advantage $\mathrm{D}(\boldsymbol{\pi}||\boldsymbol{p})$. The choice of τ indicates how much or how little these state contingent rewards and punishments matter, $\tau\{\mathrm{D}(\boldsymbol{\pi}||\boldsymbol{p}) - \mathrm{Ln}\,\frac{\pi(s)}{p(s)}\}$. In effect the augmented Log scoring rule 1.9, $m + \tau\{\mathrm{D}(\boldsymbol{\pi}||\boldsymbol{p}) + \mathrm{Ln}\,\frac{p(s)}{\pi(s)}\}$, is a way of assessing the *relative knowledge* of the elicitee, not her absolute knowledge, along with her risk tolerance. The elicitor provides an elicitee with a "benchmark" pmf $\boldsymbol{\pi}$ to predict against and rewards the elicitee according to how well she can predict relative to the benchmark's expectation of its own relative predictive advantage $\mathrm{D}(\boldsymbol{\pi}||\boldsymbol{p})$.

Why does the augmented scoring rule 1.9 elicit honest reports from risk averse forecasters? First observe that 1.9 is effectively providing the elicitee with a budget line $\boldsymbol{\pi}\bullet\boldsymbol{z}=m$, fixed income *m* to spend on contingent commodities $\boldsymbol{z}$ at prices $\boldsymbol{\pi}$. To see this note that for $\boldsymbol{z}$ satisfying 1.8, $\boldsymbol{\pi \bullet z}$= $\Sigma_s \pi(s)\{m+ \tau\{\mathrm{D}(\boldsymbol{\pi}\|\boldsymbol{p})$ +Ln $\frac{p(s)}{\pi(s)}$ }=$m+\tau\{\mathrm{D}(\boldsymbol{\pi}\|\boldsymbol{p})-\Sigma_s \pi(s)$ Ln $\frac{\pi(s)}{p(s)}$ }=m. The augmented Log scoring rule maps the simplex of coherent probability reports $\boldsymbol{p}$ into the plane {z:$\boldsymbol{\pi \bullet z}$=$m$ }. Self reporting risk tolerance and beliefs determine where on the budget line the elicitee ends up. In Figure 2,for example, the change in the Log scoring rule from ϖ(s)–$ce_Q+\tau_Q$ Ln($\frac{p(s)}{q(s)}$) to $m+ \tau\{\mathrm{D}(\boldsymbol{\pi}\|\boldsymbol{p})$ +Ln $\frac{p(s)}{\pi(s)}$ (1.8) corresponds to changing the wealth constraint facing the elicitee from Q's constraining indifference curve Yce_QX to the linear budget line constraint mEL with the elicitor's prices $\boldsymbol{\pi}$ equal to general competitive equilibrium prices $\boldsymbol{\pi}$ (any pre specified prices π will suffice since the constraining indifference curve for Q is no longer relevant).

If the elicitee is a CARA agent with beliefs ***b(s)*** and risk tolerance τ_b her expected utility maximizing choice given a budget set $\boldsymbol{\pi \bullet z}$=$m$ is to report her true beliefs ***b(s)*** and her true risk tolerance τ_b. In Figure 2, this choice occurs at point E on the budget line mEL. *Given* that the elicitee reports her true risk tolerance τ_b, any strategic report ***p(s)≠b(s)*** simply means she foregoes expected utility. It is true that honest reporting is not uniquely optimal strategy for a CARA agent with beliefs ***b(s)*** and risk tolerance τ_b. This problem arises because any particular wealth level ***z**** regarded as optimal by one CARA agent B with beliefs ***b(s)*** and risk tolerance τ_b may also be regarded as optimal by another CARA agent A with beliefs ***a(s)*** and risk tolerance τ_a. As long as the analog to 1.2 is satisfied for A, B can always pretend to be A with different beliefs and different risk attitudes and still end up with ***z****. But honest reporting *is* an optimal strategy and there is no positive advantage to pretending to be someone other than you are simultaneously on both belief and risk attitude dimensions.

■ 4 Application: the interpretation of group probability reports using Log scoring rules

The behavioural model we have developed for either the augmented or conventional Log scoring rule can provide a deep understanding of what to expect in an elicitation game with a group of possibly risk averse forecasters. To keep things simple we presume a group of two heterogeneous elicitees.

Taking the conventional Log scoring rule first, the insights of section 2 remain valid. The two risk averse elicitees should report a pmf derived from the competitive general equilibrium prices for the 3 person game involving themselves and the elicitor's agent Q. If these two forecasters are CARA agents A and B with beliefs described by pmfs ***a(s)***, ***b(s)*** and risk tolerances are τ_a and τ_b the general competitive equilibrium relative prices or *equilibrium odds* $\frac{\pi(s)}{\pi(t)}$ have a particularly simple convenient form analogous to 1.8:

1.10 $$\mathrm{Ln}\frac{\pi(s)}{\pi(t)} = \frac{\tau_a}{\tau}\mathrm{Ln}\frac{a(s)}{a(t)} + \frac{\tau_b}{\tau}\mathrm{Ln}\frac{b(s)}{b(t)} + \frac{\tau_q}{\tau}\mathrm{Ln}\frac{q(s)}{q(t)} - \frac{1}{\tau}[\varpi(s)-\varpi(t)] \text{ where } \tau=\tau_a+\tau_b+\tau_q$$

Efficiently coordinating CARA elicitees smooth their reported beliefs when faced with a conventional Log scoring rule, averaging their honest beliefs in Log-odds form by their relative risk tolerances, $\frac{\tau_a}{\tau}\mathrm{Ln}\frac{a(s)}{a(t)} + \frac{\tau_b}{\tau}\mathrm{Ln}\frac{b(s)}{b(t)} + \frac{\tau_q}{\tau}\mathrm{Ln}\frac{q(s)}{q(t)}$, and making an adjustment for the aggregate wealth at stake for themselves and the elicitor, $\frac{1}{\tau}[\varpi(s)-\varpi(t)]$. In this way they stay on their joint expected utility possibilities frontier while extracting the most they can from the elicitor.

With the augmented Log scoring rule we need to consider how an efficiently functioning pair of elicitees will act in the face of a budget constraint $\boldsymbol{\pi \bullet z}$=$m$ where $\boldsymbol{z}$ now refers to their aggregate wealth. This is not as hard a problem as it looks when the elicitees are CARA agents since the certainty equivalent function in the first equation of 1.7 satisfies the Gorman polar form for an indirect utility function and hence the sufficient conditions for exact aggregation (Cornes(1992,pp192-194) .

Define an indirect certainty equivalent (utility) function for the group by aggregating indirect certainty equivalents 1.7 for each individual facing common external contingent claims prices $\boldsymbol{\pi}$, where endowments are m_a, m_b, beliefs are $\boldsymbol{a(s)}$ and $\boldsymbol{b(s)}$ and risk attitudes are τ_a and τ_b: $m_a + m_b + \tau_a$ - D($\boldsymbol{\pi}$||$\boldsymbol{a}$)+ τ_b D($\boldsymbol{\pi}$||$\boldsymbol{b}$). The associated aggregate expenditure function for a given distribution of certainty equivalents (ce_a, ce_b) is the sum of the expenditure functions (1.4) for each agent. By 1.3 the aggregate compensated demands for the group are Z(s)= $m_a + m_b + \tau_a\{\mathrm{D}(\boldsymbol{\pi}||\boldsymbol{a}) + \mathrm{Ln}\frac{a(s)}{\pi(s)}\} + \tau_b\{\mathrm{D}(\boldsymbol{\pi}||\boldsymbol{b}) + -\mathrm{Ln}\frac{b(s)}{\pi(s)}\}$ so that aggregate wealth differences between states s and t are Z(s)–Z(t)=$\tau_a\{\mathrm{Ln}\frac{a(s)}{\pi(s)} - \mathrm{Ln}\frac{a(t)}{\pi(t)}\} + \tau_b\{\mathrm{Ln}\frac{b(s)}{\pi(s)} - \mathrm{Ln}\frac{b(t)}{\pi(t)}\}$. Rearranging this latter expression in terms of Logs of ratios $\frac{a(s)}{a(t)}$, $\frac{b(s)}{b(t)}$, and $\frac{\pi(s)}{\pi(t)}$, these aggregate wealth variations can be expressed as the differences in expenditure minimizing demands of a *single* CARA agent Z(s)–Z(t)=$(\tau_a + \tau_b)[\mathrm{Ln}\frac{c(s)}{\pi(s)} - \mathrm{Ln}\frac{c(t)}{\pi(t)}]$, where the agent has risk tolerance $\tau = \tau_a + \tau_b$ and beliefs $\boldsymbol{c(s)}$ defined in Log-odds form as $\mathrm{Ln}\frac{c(s)}{c(t)} = \frac{\tau_a}{\tau_a + \tau_b}$ - $\mathrm{Ln}\frac{a(s)}{a(t)} + \frac{\tau_b}{\tau_a + \tau_b}\mathrm{Ln}\frac{b(s)}{b(t)}$ and certainty equivalent $ce_a + ce_b$. When a group of CARA agents with diverse beliefs and risk attitudes is efficiently pooling risks, it behaves *as if* its aggregate demands in external markets are generated by a representative CARA agent whose preferences are described by a risk tolerance equal to the sum of the risk tolerances of individuals in the group,($\tau_a + \tau_b$), and beliefs , expressed as Log-odds, a weighted average of the Log-odds beliefs of members of the group, with weights equal to each agent's relative risk tolerance, $\frac{\tau_a}{\tau_a + \tau_b}\mathrm{Ln}\frac{a(s)}{a(t)} + \frac{\tau_b}{\tau_a + \tau_b}\mathrm{Ln}\frac{b(s)}{b(t)}$. By the results of section 3, the group can optimally report "honestly" the aggregated risk tolerances and risk tolerance averaged Log-odds beliefs of its representative agent when playing an augmented Log scoring rule game.

▪ 5 Limitations and conclusion

Naturally there are limitations to the augmented Log score as an elicitation process. First, the bad news. The augmented Log scoring rule can only be reasonably expected to work in situations where the elicitee can be presumed to be a CARA agent. To see why, note that the augmented scoring rule 1.9 is basically a formula for converting probability reports ***p(s)*** into affordable contingent wealth levels ***z***, ***π•z***=*m*. Any non-CARA risk averse agent trying to achieve an optimal commodity basket ***z**** on this budget line need only find a clever way of reporting beliefs ***q*** and risk tolerances τ to achieve her desired target. How? Equation 1.11, which is 1.2 expressed in relative price form at prices ***π*** and evaluated at ***z**** , must be satisfied by any CARA agent with beliefs ***q*** and risk tolerance τ who would make z* her optimal choice.

$$1.11 \qquad \frac{\pi(s)}{\pi(t)} = \frac{q(s)}{q(t)}\, e^{-\frac{1}{\tau}\,(z^*(s)-z^*(t))}$$

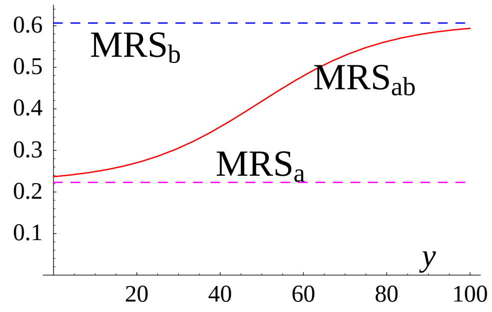
For any pair of states *s*,*t* and *any* fixed τ, the required subjective odds $\frac{q(s)}{q(t)}$ can be found by solving 1.11 in terms of the given relative prices $\frac{\pi(s)}{\pi(t)}$ and specified wealth differences *z*(s)–z*(t)*. Acting as if she is a CARA agent reporting the pmf ***q*** determined by these odds $\frac{q(s)}{q(t)}$ and the arbitrarily chosen τ will ensure that her optimal choice ***z**** is realized.The bad news is that these strategically chosen beliefs need bear no obvious relation to the agent's honest beliefs.

However, there is some good news to go along with the bad. An agent with beliefs ***q*** whose risk attitude is represented by a finite mixture of CARA agents can in some cases find a judiciously chosen CARA agent with the same beliefs ***q*** that can adequately, and honestly, represent or "approximate" their interests (***z****). This is indeed good news because Brockett and Golden (1987) have shown that all "proper" utility of wealth functions -where "proper" now means smooth utility of wealth functions with derivatives that start positive and alternate in sign - are mixtures (not necessarily finite) of CARA agents and all risk averse utility of wealth functions can be approximated by finite mixtures of CARA agents.

Figure 3 presents an illustrative example for two states. Consider an agent with a utility of wealth function u_{ab}(z)=$-ae^{-\frac{1}{\tau_a}z}$ $-be^{-\frac{1}{\tau_b}z}$, a finite mixture in proportions $\frac{a}{b}$ of two CARA utility of wealth functions with risk tolerances $\tau_a < \tau_b$. For any given wealth difference *d*=z(1)–z(0) in two states sϵ{0,1} the mixture agent's marginal rate of substitution $\text{MRS}_{ab} = \frac{q(1)}{q(0)} \frac{u'_{\text{ab}}(y+d)}{u'_{\text{ab}}(y)}$ at the wealth risk (*y*+*d*,*y*) for some level of sure wealth *y*, is increasing in *y*, roughly from the constant $\text{MRS}_a = \frac{q(1)}{q(0)} e^{-\frac{d}{\tau_a}}$ for the lower risk tolerant agent to $\text{MRS}_b = \frac{q(1)}{q(0)} e^{-\frac{d}{\tau_b}}$ for the more risk tolerant agent. If *d* is the difference in wealth z*(1)-z*(0) for the mixture agent at her optimal choice ***z****, where $\text{MRS}_{ab} = \frac{\pi(1)}{\pi(0)}$ for optimality, then continuity ensures there will be some intermediate value of risk tolerance τ_c such that $\text{MRS}_c = \frac{q(1)}{q(0)} e^{-\frac{d}{\tau_c}} = \frac{\pi(1)}{\pi(0)} = \text{MRS}_{ab}$. The CARA agent with beliefs ***q*** and risk tolerance τ_c will make the same choice ***z**** in the budget set as the mixture agent. The mixture agent can maximize her expected utility given the augmented Log scoring rule by reporting her beliefs honestly and the calculated risk tolerance τ_c .The risk tolerance report τ_c is not strictly "honest ", but if honesty is asked for then a scoring rule needs to be devised that permits an agent to report a risk tolerance *function*, along with her beliefs, not a single number.

Figure 3

Parameters: $d=15, \tau_a=10, \tau_b=10, \frac{q(1)}{q(0)}=1, a=0.9, b=0.1$

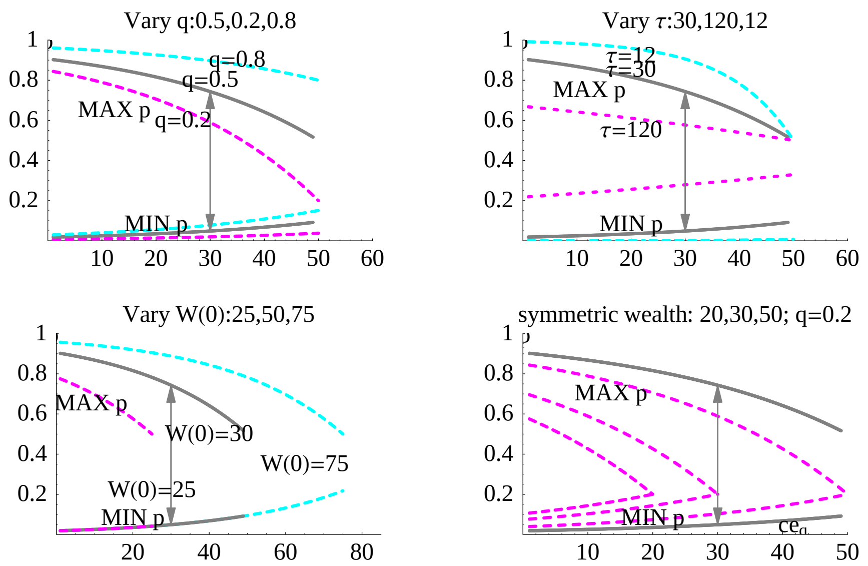


Another limitation is that the range of beliefs that can be credibly elicited in either the conventional or augmented Log scoring rule game depends critically on the elicitor's choice of aggregate wealth endowment $\varpi(s)$ and parameters for the constraining residual wealth functions (q ce_Q, and τ_Q in the conventional game, π and m in the augmented game). Take the conventional Log scoring rule as an example. Referring back to Figure 1, the range of tangent lines to the constraining indifference curve defining the elicitee's residual wealth is definitely influenced by the choice of Q's certainty equivalent ce_Q. The more wealth the elicitor keeps for himself (higher ce_Q shifts the residual wealth constraint closer to the origin for the elicitee) the smaller the range of prices that can be elicited without running into a corner solution. Once a corner is reached the elicitation experiment will suffer from extreme forms of the "flat maximum" problem (von Winterfeldt and Edwards, pp 420-426) - the elicitee's payoff (in expected wealth or certainty equivalent terms) won't change by "stretching" the reported price beyond that associated with the corner solution. Figure 4 plots the minimum and maximum elicitable probabilities p for S=1 as a function of the level of ce_Q for various combinations of experimentally determined parameters (changing beliefs q=q(1), varying Q's risk tolerance τ, varying wealth in one state only or in both states symmetrically). In each diagram the grey bar with the arrows indicates the maximum and minimum ranges for ce_Q=30 as in Figure 1, and corresponding upper and lower bounding curves (a colour coordinated version of Figure 4 is available from the author on request).

Figure 4 is of course only illustrative , but it shows that if there is some public, shared prior knowledge of "ballpark" beliefs of the elicitee, the elicitor may be able, through a judicous choice of experimental parameters, to design a credible experiment at low expected utility loss to himself and simulataneously positive benefit to the elicitee. The augmented Log scoring rule 1.9 may be especially useful in the case of eliciting expert opinion for commonly but imprecisley known small probability events since the elicitor doesn't have to be able to predict the event well (ie with tiny probabilities necessitating large payoffs to induce particpation) but only *relatively* better than the benchmark π assessed by the elicitor.

In conclusion we note, with Savage (1971) , that being able to honestly elicit beliefs from CARA agents, while not completely general, is an improvement over having to assume risk neutrality. Perhaps the most helpful insights in the paper however are derived from explicit behavioural models of what to expect when risk averse elcitees, alone or in groups, face an explicit incentive mechanism like the Log scoring rule. As Savage said, personal probabilities are indeed prices, but only in a manner of speaking. Everything hinges on the manner of speaking.

Figure 4



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