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The Nucleolus Strikes Back*

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ABSTRACT

In a recent paper in *Decision Sciences*, Barton proposes the MTPD method for finding a unique solution for the nucleolus. We argue that the basis for the MTPD method is flawed because the nucleolus is always unique. Furthermore, the MTPD method is inconsistent and does not exhibit the properties desirable in a cost allocation method.

Subject Areas: Computer Applications, Cost Allocations, Game Theory, and Managerial Accounting.

INTRODUCTION

The nucleolus has been proposed as a method for allocating joint costs among multiple entities sharing a common resource [5] [11] [12]. In a recent issue of this journal, Barton claims to identify "a problem in applying the nucleolus for allocating joint costs not envisioned in previous research" [1, p. 366]. The alleged problem is that the nucleolus may result in multiple optimal solutions. Barton proposes a solution to this problem by applying a "related alternative method," the minimum total propensity to disrupt (MTPD) method, to select one of the multiple solutions. Subsequently, Grange and Kochenberger [4] implemented the MTPD in a spreadsheet. In this paper, we demonstrate that the basis for the MTPD method is flawed. Properly applied, the nucleolus always consists of a unique allocation (see [10], Theorem 2). Moreover, compared to the nucleolus, the MTPD method has little to recommend it, either conceptually or computationally.

The Cost Allocation Problem

Consider a cost allocation game comprising four entities (A, B, C, and D), which utilize a common resource [1]. The total costs incurred by various combinations (coalitions) of the entities are listed in Table 1. To model as a game, we define the characteristic function as the cost savings to each coalition of cooperating, rather than each acting individually. That is:

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$$v(S) = \sum_{i \in S} c_i - c(S),$$

where c_i is the cost to entity i of purchasing individually and $c(S)$ is the aggregate cost to coalition S of purchasing jointly, as listed in the third column of Table 1. The cost savings ($v(S)$) are listed in the last column.

A solution to the cost allocation game is an allocation x of the total cost savings $v(\{A,B,C,D\}) = 636.60$ to the participants (players) in the game, which is a vector (x_A, x_B, x_C, x_D) such that $x_A+x_B+x_C+x_D = \$636.60$. Any allocation x of the cost savings implies a distribution of the total costs among the four participants according to the formula:

$$\text{Cost share of } i = c_i - x_i.$$

Clearly, not every allocation x of the cost savings will be equally acceptable to all the participants. In particular, no individual or subgroup is likely to accept an allocation of the total cost savings that is less than they could achieve by unilateral action independent of the rest of the participants. A minimal requirement for a cost allocation would appear to be that everyone gains from cooperation, that is:

$$\sum_{i \in S} x_i \geq v(S) \qquad \forall S \subseteq N, \tag{1}$$

where $N = \{A,B,C,D\}$. The set of allocations that satisfy this requirement is called the "core" of the game.

The core of this cost allocation problem is illustrated in Figure 1. The wire frame (tetrahedron) depicts the set of all possible allocations of cost savings among the four entities. The core is the shaded shape in the interior of the tetrahedron. Typically, as in this example, there are many allocations in the core. A cost allocation method is required to select a particular allocation from the core to serve as the solution of the game. The nucleolus is one such solution; the MTPD method is another. Indeed, a plethora of alternative allocation procedures have been advocated. One advantage of a game-theoretic approach to the problem is that it enables the properties of different proposals to be compared.

In some games, there may be no core allocations, that is, the core may be empty. However, cost allocation games are typically convex, which implies that core allocations always exist. For an introduction to cooperative game theory, in general, and cost allocation games, in particular, see [2].

THE NUCLEOLUS

The nucleolus as a solution concept for cooperative games was introduced by Schmeidler [10] specifically to overcome the multiplicity of outcomes characteristic of its antecedent concepts, the bargaining set and the kernel. An allocation in the nucleolus minimizes the potential dissatisfaction of any coalition with its share of the total cost savings. We now make this precise, using a new characterization of

Table 1: Basic information for coalitions.

Coalition	Quantity	Total Cost $c(S)$	Average Cost	Coalition Savings $v(S)$
A	100	200.00	2.00	
B	300	431.54	1.44	
C	500	617.04	1.23	
D	700	780.91	1.12	
AB	400	527.81	1.32	103.73
AC	600	701.03	1.17	116.01
AD	800	857.42	1.07	123.49
BC	800	857.42	1.07	191.15
BD	1000	1002.38	1.00	210.07
CD	1200	1138.83	0.95	259.12
ABC	900	931.11	1.03	317.46
ABD	1100	1071.54	0.97	340.91
ACD	1300	1204.46	0.93	393.49
BCD	1500	1331.36	0.89	498.12
ABCD	1600	1392.89	0.87	636.60

the nucleolus [9], which clarifies its appeal as a method for minimizing potential discordance in the grand coalition.

The potential dissatisfaction of a coalition with an allocation x can be measured by its "excess," which is the difference between the cost savings it enjoys at the allocation x and the cost savings it could obtain by acting alone:

$$e(x,S) = v(S) - x(S).$$

This can be expressed as:

$$e(x,S) = \sum_{i \in S}(c_i - x_i) - c(S).$$

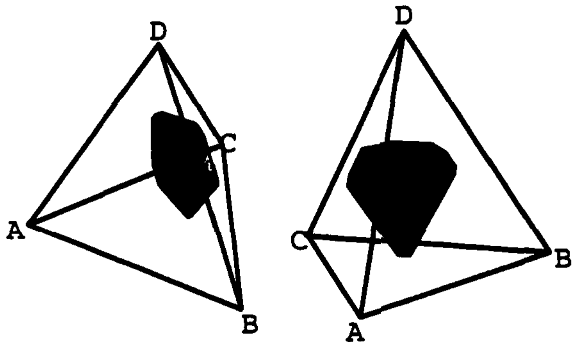
The first term is the total cost share of the coalition S under the proposed allocation. The second term is the total costs they would bear if they acted unilaterally. The smaller the excess, the more satisfied is the coalition at the allocation x. An allocation belongs to the core if and only if $e(x,S) \leq 0$ for all coalitions S. That is, in the core, joint action is better than unilateral action for every coalition.

Within the core, the excess provides a measure of the degree to which the coalition is sharing the benefits of cooperation. An allocation y is more favorable to coalition S than an allocation x whenever:

$$e(y,S) < e(x,S).$$

A coalition S has an "objection" to x if there exists another allocation y that is more favorable to S, that is, $e(y,S) < e(x,S)$. Their objection has a "counterobjection" if there

Figure 1: The core of the cost allocation problem: Two views.



exists another coalition T that is worse off at y and, furthermore, whose dissatisfaction (excess) with y is greater than coalition S's dissatisfaction with x. That is, T has a counterobjection to y if $e(y,T)>e(x,T)$ and $e(y,T)\geq e(x,S)$. The nucleolus Nu is the set of all allocations x with the property that for every objection (y,S) there is a counterobjection. It is the set of allocations to which no group can validly object. This is the basis for the claim that the nucleolus is a reasonable solution to the cost allocation problem.

Surprisingly, in any game, there is always precisely one allocation to which there is no valid objection. That is, the nucleolus is non-empty and comprises a single point (see Theorems 1 and 2 in [10] and Proposition 288.4 in [9]). Barton's "problem" is nonexistent. Indeed, as Schmeidler remarked in his introduction, uniqueness is "one of the most appealing properties of the nucleolus as a solution concept" [10, p. 1164].

COMPUTING THE NUCLEOLUS

Typically, algorithms for computing the nucleolus proceed by solving a sequence of minimization problems [7]. Recall that the nucleolus is the set of allocations to which no group can object. The ability of a coalition S to object to an allocation x depends upon the excess $e(x,S)$, which measures the potential dissatisfaction of the coalition with the allocation x. A useful first step is to identify those allocations that are least objectionable, that is, those allocations that have the smallest excess.

Consider the following linear program:

$$\min_{x\in X} r, \tag{2}$$

such that

$$e(x,S)\leq r \qquad \forall S\subseteq N,$$

where X is the set of feasible allocations,

$$X = \{(x_A, x_B, x_C, x_D) \in \Re^4 : x_i \geq 0$$

and

$$x_A + x_B + x_C + x_D = v(N)\}.$$

(X is depicted by the wire frame in Figure 1.) The solution to this linear program comprises a minimum value r^1 and a set of allocations at which this minimum value is obtained. Let X^1 denote the set of optimal solutions to the linear program. X^1 is the set of allocations that minimizes the excess (potential dissatisfaction) over all coalitions. The greater the excess, the greater the incentive for a coalition to object. In this sense, the set X^1 contains those allocations that are least objectionable. It is known as the "least core."

Solving the preceding linear program for the game in [1], the least core is given by the following system of equalities and inequalities:

$$x_A = 69.24,$$

$$103.73 \leq x_B \leq 173.87,$$

$$116.01 \leq x_C \leq 226.45,$$

$$123.49 \leq x_D \leq 249.90,$$

$$x_B + x_C + x_D = 567.36, \tag{3}$$

which lies in a plane perpendicular to the A axis in the three-dimensional simplex (see Figure 2).

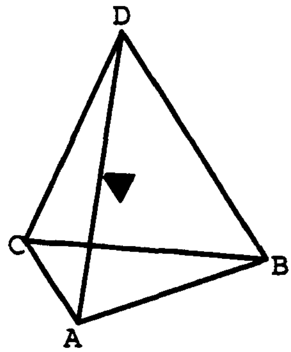
In the least core, the payoff to player A is fixed at 69.24. In other words, player A cannot validly lay claim to a larger share of the cost savings. Any such claim could be met by a justifiable objection from the other players. By the same token, player A would have a justifiable objection against any allocation that awarded him a smaller share. Any unobjectional allocation must award player A exactly 69.24. After satisfying A, there remains some flexibility in the shares awarded to the other players. While the total payoff to the other players is constrained to:

$$x_B + x_C + x_D = 567.36,$$

their individual shares may vary within the bounds given by the inequalities in (3). Even within this range, there are allocations to which some of the players may justifiably object. For example, consider the allocation (69.24, 173.87, 192.99, 200.50), which allocates the maximum possible of the remaining gains to player B (173.87) and shares the rest (over and above their minima) equally between C and D. The coalition $\{C,D\}$ can justifiably object with the amended allocation (69.24, 146.25, 198.83, 222.28), to which B has no counterobjection.

Typically, as in this example, the least core comprises a subset of the core allocations (assuming the core is non-empty), which are unobjectionable for some coalitions. But it also contains allocations that are open to objection by other coalitions.

Figure 2: The least core in Barton's game.



To isolate the nucleolus, we now look for the least objectionable allocations in the least core. To do this, we solve a second linear program:

$$\min_{x \in X^1} r, \tag{4}$$

such that

$$e(x,S) \leq r \qquad \forall S \in \mathcal{B}^1,$$

where $\mathcal{B}^1$ comprises those coalitions that have valid objections to points in X^1. In this case, the program has a unique solution:

$$x^N = (69.24, 146.25, 198.83, 222.28),$$

which must be the nucleolus of the game. It is the only allocation to which no coalition can justifiably object.

If the preceding linear program (4) did not lead to a unique solution, we would repeat the procedure. In the Appendix, we demonstrate that this iterative procedure eventually terminates in a unique solution that is the nucleolus of the game. Thus, the nucleolus can be computed by solving a sequence of linear programs. A *Mathematica* implementation of this algorithm is given in [2].

The MTPD Method

Apparently, in proposing the MTPD method, Barton [1] confuses the least core with the nucleolus. Faced with a multiplicity of acceptable solutions in this first round, he surrenders the previous objective of minimizing objections, substituting the alternative objective of minimizing the nonlinear function $a/x^A + b/x^B + c/x^C + d/x^D$, where a, b, c and d are constants related to the contributions of the coalitions to total savings. In short, the MTPD method substitutes the nonlinear program:

$$\min_{x \in X^1} \sum_{i \in N} \frac{a_i}{x_i}, \tag{5}$$

for the linear program (4). As a convex function on a convex domain, this problem is guaranteed to have a unique solution.

Although the nucleolus does in fact provide a unique allocation, does the MTPD method have any merit as a cost allocation solution in its own right? We believe not. We raise three fundamental conceptual objections to MTPD:

- it is inconsistent;
- the second objective function is ad hoc, yielding a solution without desirable properties or rationalization;
- it makes only selective use of the available information in the second round.

The MTPD Method Is Inconsistent

Consistency is widely regarded as one of the most desirable properties of any solution concept in a cooperative game [6]. Consistency requires that the solution to an appropriately defined "reduced game" yield the same allocations as those specified in the original game. Consequently, no subset of players will have any incentive to defect and play "their own game." Consistency of the nucleolus is evident in the preceding algorithm, where the same objective (minimizing excess) is applied at each stage. In Barton's example, the nucleolus of the reduced game in which player A's allocation is held fixed at 69.24 specifies allocations of 146.25, 198.83 and 222.28 to B, C, and D, respectively, which is precisely their allocations in the nucleolus of the original game.

The MTPD method applies one objective (minimizing objections) to the whole game (which determines A's allocation), and then applies a different objective (minimizing total potential to disrupt) to the reduced game. To illustrate the significance of this distinction, imagine that an additional player E is added to Barton's game in such a way that the least core of the new game corresponds with the original four-player game. Being consistent, the nucleolus would specify the same allocations to A, B, C and D in the new game as in the original game. The MTPD method applied to the new game would yield the allocation (120.02, 159.02, 175.37, 182.19) to A, B, C and D, respectively. This is an allocation that is specifically rejected [1, pp. 370-371] as unsuitable in the original game.

The Objective Function Is Ad Hoc

The MTPD method's starting point is a proposal by Gately [3], who suggested equalizing the propensities to disrupt as a practical solution for a real cost allocation problem. Gately argued that a coalition S would have a propensity to disrupt an allocation x if its share of the gains from cooperation were small relative to those of the remaining players. Accordingly, he defined the propensity of a coalition S to disrupt an allocation x as the ratio of the excesses of the coalition and its opponents at x, that is:

$$d(x,S) = \frac{e(x,S^c)}{e(x,S)},$$

where S^c is the complement of S. For individual players, this can be equivalently expressed as:

$$d(x,i) = \frac{\varphi_i}{x_i} - 1,$$

where

$$\begin{aligned}\varphi_i &= v(N) - v(N-i) \\ &= c_i - (c(N) - c(N-i))\end{aligned} \tag{6}$$

is player i's (marginal) contribution to total cost savings. Gately suggested that an allocation at which the individual propensities to disrupt were equalized would be an acceptable solution to the cost-sharing problem. It is easily shown that such a point exists and is achieved where the maximum propensity to disrupt is minimized. At this point, total cost savings are shared in proportion to marginal contributions, that is:

$$x_i^{PD} = \frac{\varphi_i}{\sum_{i \in N} \varphi_i} v(N). \tag{7}$$

This is, in fact, the allocation obtained under the separable costs-remaining benefit (SCRB) method [12]. This method has an impressive practical pedigree—it is popular in the engineering literature and regularly used for cost allocation in multipurpose reservoir projects in the United States and other countries [13].

Instead of selecting an allocation that minimizes the maximum individual propensity to disrupt (which has an obvious affinity with the nucleolus), the MTPD method minimizes the aggregate propensity to disrupt. That is, instead of solving the problem:

$$\min_{x \in X^1} \max_{i \in S} \frac{\varphi_i}{x_i} - 1, \tag{8}$$

MTPD solves the problem:

$$\min_{x \in X^1} \sum_{i \in N} \frac{\varphi_i}{x_i} - n, \tag{9}$$

where X^1 is the least core and n is the number of entities. Barton [1] offers two reasons for favouring the total over the individual propensity to disrupt, namely that the total propensity to disrupt:

- offers an overall measure of discordance, and
- can be expressed as a function of x_i having a unique minimum value.

The second reason can be readily dismissed. The individual propensity to disrupt (8) also yields a unique minimum value, as does the nucleolus and a host of other possibilities. Uniqueness offers no grounds for choosing one objective over another. The unique solution to (8) in Barton's example is the allocation (69.24, 160.77, 195.54, 211.05).

The first reason is no more persuasive. The whole point of minimizing dissatisfaction or propensity to disrupt is to ensure the stability of an allocation against defection by disgruntled entities. Why should potential defectors pay any attention to the aggregate propensity to disrupt, rather than their individual propensity to disrupt, when considering whether to defect? Aggregation seems singularly inappropriate for the purpose at hand. It implies that a particular entity will be more likely to defect if its discontent is shared by the other entities. On the contrary, we would argue that an entity is more likely to defect if its propensity to disrupt is disproportionate.

Furthermore, if aggregation is appropriate, why minimize the aggregate excess in the first round? Why not replace the first stage minimization (2) with the alternative:

$$\min_{x \in X} \sum_{S \subseteq N} e(x,S),$$

which could be justified on exactly the same grounds as replacing (8) with (9)? The answer is simple—it is the disatisfaction of particular coalitions, and not the aggregate disatisfaction, which fuels potential defection.

The Method Ignores Relevant Information

The advantage of an allocation method based on a game theoretic solution (such as the nucleolus or the Shapley value) is that it incorporates in some fashion information regarding all the alternatives available to the participants. Allocation is based on all the 2^n numbers, which constitute the characteristic function of the game, rather than just a subset of n numbers as in the popular SCRB method (7).

The MTPD method uses all the information in computing the least core, but then relies solely on the marginal contributions φ_i in selecting a particular allocation from the least core (9). Barton recognises that it would be inappropriate to apply (9) to the original game, writing:

> Actually, the nucleolus is a much better stand-alone method than the MTPD. The MTPD has a very serious deficiency as a stand-alone method: It looks at the problem from the viewpoint of the set of last entitities added to coalitions that contain (n–1) members. [1, p. 370]

To illustrate, he shows that total propensity to disrupt is minimized over all allocations at x^{MTPD} = (120.02, 159.02, 175.37, 182.19). Barton argues that this is probably unsatisfactory, because the share awarded to the three largest entities is too small relative to that awarded to A. The coalition comprising B, C and D is liable to object to this allocation, since they receive only 13 percent of the gains from A's participation, compared to A's 87%. In Gately's terms [3], the coalition comprising the three largest entities $\{B,C,D\}$ has a high propensity (6.5) to disrupt this allocation, since their excess at x^{MTPD} of 18.46 is very small relative to A's excess of 120.02.

Barton goes on to claim that this problem will not arise in his method, since it is constrained to select an allocation from the least core. However, this misses the real point. As the number of players increases, the dimension of the least core can grow proportionately. As the dimension of the least core increases, the number of possible coalitions and strength of possible objections to extreme allocations within the least core will also increase. Barton's objections to applying his method to the overall game will apply with increasing strength to the MTPD method as the number of players increases.

For this reason, the MTPD method seems designed with four-player games in mind. It is unnecessary in three-player games, since the midpoint of the least core yields the nucleolus. Its application to games with more than four players seems destined, for the reasons just discussed, to incur the very problems it is designed to overcome.

Computational Efficiency

Does the MTPD method offer any computational advantage? We think not. The computational burden in finding the nucleolus involves identifying the bounds of the least core. MTPD must do this also. In the second stage, a constrained nonlinear program (5) is substituted for the linear program (4). In general, this optimization problem could be rather difficult to solve. By comparison, the linear program (4) is straightforward and easy to solve. On the other hand, the MTPD solution terminates at this point, whereas the nucleolus might require further iterations. Computationally, the trade-off is a more complicated optimization problem at the second round against possibly fewer iterations. Given the efficiency of linear programming algorithms, we think it unlikely that the MTPD method will have a computational advantage.

CONCLUSION

In this paper, we have analysed the MTPD method for finding a unique solution in cost allocation problems. We have argued that it is a poor solution to a non-existent problem. The problem is non-existent because the nucleolus as defined by Schmeidler [10] is unique and widely accepted as a cost allocation method [11] [12] [13]. Moreoever, the proposed solution seems arbitrary and inconsistent, and does not exhibit the properties desirable in a cost allocation method. [Received: September 2, 1994. Accepted: September 22, 1995.]

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APPENDIX
Computing the Nucleolus

Let Nu be the nucleolus of a game (N,v) with player set N and characteristic function v. In this appendix, we demonstrate that the algorithm outlined in our paper will compute the nucleolus Nu. The least core is the set of optimal solutions to the linear program:

$$\min_{x \in X} r,$$

such that

$$e(x,S) \leq r \qquad \forall S \in \mathcal{N},$$

where X is the set of feasible allocations and $\mathcal{N}$ is the set of proper coalitions; that is:

$$\mathcal{N} = \{S \subset N : S \neq \emptyset, N\}.$$

Denote the solution set X^1 and the minimal value r^1.

By construction, there is no coalition and no allocation in X^1 at which the excess exceeds r^1. There are some coalitions for which the excess is constant and equal to r^1 for all allocations $x \in X^1$. Let $\mathcal{A}^1$ denote the class of such coalitions. The remaining coalitions $\mathcal{B}^1$ must have some point in X^1 at which their excess is less than r^1. That is, the proper coalitions fall into two classes defined as follows:

$$\mathcal{A}^1 = \{S \in \mathcal{N}: e(x,S) = r^1 \qquad \forall x \in X^1\},$$

$$\mathcal{B}^1 = \{S \in \mathcal{N}: e(x,S) < r^1 \qquad \text{for some } x \in X^1\}.$$

Coalitions in $\mathcal{A}^1$ can neither object to any allocation in X^1 nor counterobject to any objection in X^1. By restricting attention to allocations in X^1 (the least-objectionable allocations), their bargaining power has been exhausted and they are effectively neutralized. If all coalitions belong to $\mathcal{A}^1(\mathcal{B}^1{=}\emptyset)$, then the least core X^1 contains a single point to which there are no valid objections. We have located the nucleolus.

If not, we simply repeat the preceding minimization with respect to those coalitions that still have the potential to object (that is, $\mathcal{B}^1$). Formally, we solve the following linear program:

$$\min_{x \in X^1} r,$$

such that

$$e(x,S) \leq r^1 \qquad \forall S \in \mathcal{B}^1,$$

where X^1 is defined as:

$$X^1 = \{x \in X : \text{such that } e(x,S) = r^1 \qquad \forall S \in \mathcal{A}^1,$$

$$e(x,S) \leq r^1 \qquad \forall S \in \mathcal{B}^1\}.$$

Let X^2 and r^2 be the set of optimal solutions and the optimal value, respectively. As before, coalitions fall into two classes:

$$\mathcal{A}^2 = \{S \in \mathcal{B}^1 : e(x,S) = r^2 \qquad \forall x \in X^2\}$$

$$\mathcal{B}^2 = \{S \in \mathcal{B}^1 : e(x,S) < r^2 \qquad \text{for some } x \in X^2\}$$

$$= \mathcal{B}^1 / \mathcal{A}^2.$$

Furthermore, $X^2 \subseteq X^1$, $\mathcal{A}^2 \neq \emptyset$ and, consequently, $\mathcal{B}^2 \subset \mathcal{B}^1$.

Continuing in this fashion, we solve a sequence of linear programs:

$$\min_{x \in X^{i-1}} r,$$

such that

$$e(x,S) \le r^{i-1} \qquad \forall S \in \mathcal{B}^{i-1},$$

$$i = 1, 2, ..., k, \tag{A1}$$

with optimal solutions X^i and values r^i. At each stage, the coalitions $S \in \mathcal{B}^{i-1}$ are partitioned into two classes:

$$\mathcal{A}^i = \{S \in \mathcal{B}^{i-1} : e(x,S) = r^i \qquad \forall x \in X^{i-1}\}$$

$$\mathcal{B}^i = \{S \in \mathcal{B}^{i-1} : e(x,S) < r^i \qquad \text{for some } x \in X^{i-1}\}$$

$$= \mathcal{B}^{i-1} / \mathcal{A}^i.$$

Clearly, $X^i \subseteq X^{i+1}$, since X^{i-1} is the feasible set and X^i the solution set of the *i*th linear program. Similarly, $r^i \le r^{i-1}$. By construction, $\mathcal{B}^i \subseteq \mathcal{B}^{i-1}$. In the following proposition (adapted from Lemma 6.3 of [8]), we demonstrate that the nucleolus belongs to X^i for every i and $\mathcal{B}^i \subset \mathcal{B}^{i-1}$. Since $\mathcal{B}$ is finite, the algorithm terminates when $\mathcal{B}^k = \emptyset$. At this stage, X^k contains a single element, which must be the nucleolus of the game.

Proposition 1: Let $X^i, \mathcal{A}^i, \mathcal{B}^i$ be defined by the sequence of linear programs (A1). Then:

1. $\text{Nu} \subseteq X_i$, $i = 1, 2 \ldots$
2. $\mathcal{B}^i \subset \mathcal{B}^{i-1}$, $i = 1, 2 \ldots$
3. $\text{Nu} = X_i \Leftrightarrow \mathcal{B}^i = \emptyset$.

Proof.

1. For any i, let x be any allocation not in X^i. We show there exists an allocation in X^i that can be used to object to x. That is, there exists a coalition S and allocation $y \in X^i$ such that:

$$e(y,S) \le r^i < e(x,S),$$

since y is a solution of the linear program while x is not. Furthermore, there is no counterobjection to y. Every coalition T that is worse off at $y(e(y,T) > e(x,T))$ is, however, better off at y than S was at x, since:

$$e(y,T) \le r^i < e(x,S).$$

We have shown that for every $x \notin X^i$ there exists a $y \in X^i$ that provides an unassailable objection to y. Since the nucleolus is unobjectionable, $\text{Nu} \subseteq X_i$.

2. Enumerate the coalitions in $\mathcal{B}^i$ as:

$$\mathcal{B}^i = \{S^1, S^2, ..., S^m\}.$$

For every coalition in $\mathcal{B}^i$, there exists some allocation $x^j \in X^i$ with $e(x^j, S^j) < r^i$. Let $\bar{x}$ be the average of these allocations, that is:

$$\bar{x} = \frac{1}{m} \sum_{j=1}^{m} x^j.$$

Since X^i is convex, $\bar{x} \in X^i$. By the fundamental theorem of linear programming, there exists some $\bar{S} \in \mathcal{B}^{i-1}$ with:

$$e(\bar{x}, \bar{S}) = r^i. \tag{A2}$$

Also, since $x^j \in X^i$;

$$e(x^j, \bar{S}) \leq r^i, \qquad j = 1, 2, ..., m.$$

We claim that $\bar{S} \notin \mathcal{B}^i$. Suppose otherwise; that is, suppose that $\bar{S} = S^l$ for some $1 \leq l \leq m$. Then:

$$e(x^l, \bar{S}) < r^i,$$

which implies:

$$e(\bar{x}, \bar{S}) = \frac{1}{m} \sum_{j=1}^{m} e(x^j, \bar{S}) < r^i.$$

This contradicts (A2). We conclude, therefore, that $\bar{S} \in \mathcal{B}^{i-1} / \mathcal{B}^i$ and

$$\mathcal{B}^i \subset \mathcal{B}^{i-1}.$$

3. Let $x \in X_i$. If $\mathcal{B}^i = \emptyset$, then there exists no coalition S and no allocation $y \in X_i$ for which $e(y, S) < r_i$. Therefore, there is no objection to x, so that $x \in$ Nu.

   Conversely, assume X^i contains a single element, that is $X^i = \{x\}$. By part 1, X^i=Nu. Since x is a solution to the ith linear program, $e(x, S) = r^i$ for every $S \in \mathcal{B}^{i-1}$. Therefore, $\mathcal{B}^i = \emptyset$.

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